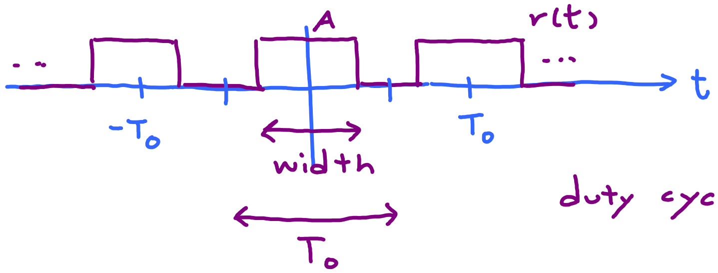
Q1 Square Wave: Fourier coefficients and duty cycle

Wednesday, October 21, 2015 5:54 PM



duty cycle = $\frac{\text{width}}{T_0}$

(a) In class, we've seen that when the duty cycle is $\frac{1}{n}$, the n^{th} harmonic (along with its multiples) is suppressed. Here, $c_4 = 0$. So, we conclude that the duty cycle is

$\frac{1}{4} = 25\%$.

(b) Recall that $c_k = \frac{1}{T_0} \int_{T_0} r(t) e^{-j2\pi k f_0 t} dt$.

Therefore, $c_0 = \frac{1}{T_0} \int_{T_0} r(t)\, dt = \langle r(t) \rangle$ ← average.

From the picture, $\langle r(t) \rangle = \frac{\text{width} \times A}{T_0} = (\text{duty cycle}) \times A$.

Therefore, $A = \frac{\langle r(t) \rangle}{\text{duty cycle}}$

We are given that $c_0 = \frac{1}{2}$ and we found, in part (a), that duty cycle $= \frac{1}{4}$.

Therefore, $A = \frac{1/2}{1/4} = 2$.

Q2 Powered Cosine Modulator

Sunday, July 03, 2011 6:01 PM

(a) First, we use the product-to-sum formula

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

to expand $\cos^3 x$ into sum of weighted $\cos(kx)$.

$$\cos^2 x = \cos x \cos x = \frac{1}{2}\left(\cos(2x) + \cos(0)\right) = \frac{1}{2}\left(\cos 2x + 1\right)$$

$$\cos^3 x = \cos x \cos^2 x = \cos x \left(\frac{1}{2}\left(\cos 2x + 1\right)\right)$$

$$= \frac{1}{2}\left(\underbrace{\cos x \cos 2x}_{= \frac{1}{2}\left(\cos(3x) + \cos x\right)} + \cos x\right) = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

Plugging in $x = \omega_c t = 2\pi f_c t$, we get $\cos^3 \omega_c t = \frac{1}{4}\cos(3\omega_c t) + \frac{3}{4}\cos(\omega_c t)$.

At point (c), we want $k\, m(t) \cos \omega_c t$

At point (b), we have $m(t)\cos^3 \omega_c t = \underbrace{\frac{1}{4} m(t)\cos(3\omega_c t)}_{\text{don't want this part}} + \underbrace{\frac{3}{4} m(t)\cos(\omega_c t)}_{\text{want this part}}$.

Any bandpass filter centered at $\pm f_c$ will work.

In addition, the passband of this filter must be larger than $2B$.

Note that if the gain of the BPF is 1, then $k = \frac{3}{4}$.

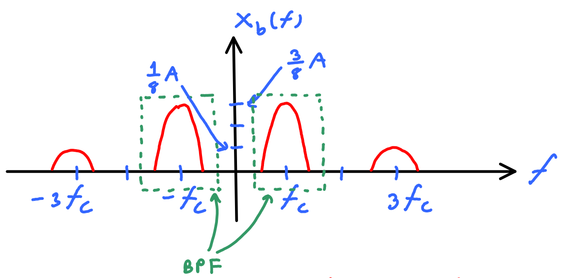
(b)

(b.1) Let $x_b(t)$ be the signal at point (b).

Then $x_b(t) = m(t)\cos^3 \omega_c t = \frac{1}{4} m(t)\cos(3\omega_c t) + \frac{3}{4} m(t)\cos(\omega_c t)$

$$\xrightarrow{\mathcal{F}} \frac{1}{8} M(f - 3f_c) + \frac{1}{8} M(f + 3f_c) + \frac{3}{8} M(f - f_c) + \frac{3}{8} M(f + f_c)$$

where $f_c = \omega_c / 2\pi$.



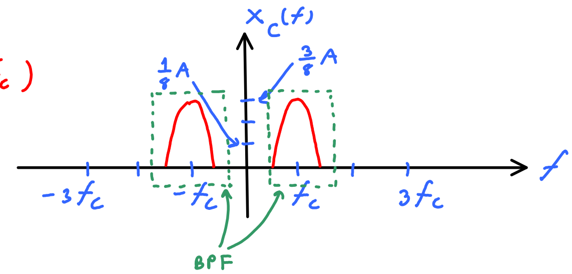
The frequency bands occupied are $[-3f_c - B, -3f_c + B]$,
$[-f_c - B, -f_c + B]$,
$[f_c - B, f_c + B]$, and
$[3f_c - B, 3f_c + B]$.

(b.2) Let $x_c(t)$ be the signal at point (c).

We will assume that the gain of the BPF is 1. (In general, if gain $= g$, then $k = \frac{3}{4} g$)

In which case, $x_c(t) = \frac{3}{4} m(t) \cos \omega_c t$

and $X_c(f) = \frac{3}{8} M(f - f_c) + \frac{3}{8} M(f + f_c)$



The frequency bands occupied are $[-f_c - B, -f_c + B]$ and $[f_c - B, f_c + B]$

(c) To avoid overlapping of spectra at point (b),

we must have $f_c - B > 0$, and $f_c + B < 3f_c - B$.

Both conditions require $f_c > B$.

Hence, the mininum usable value of f_c is B.

(d) Recall (from part (a)) that $\cos^2 \omega_c t = \frac{1}{2} + \frac{1}{2} \cos(2\omega_c t)$.

There is no component around f_c. Hence, this system would not give the desired output.

(e) As in part (a), we need to expand $\cos^n(x)$ into a linear combination of $\cos(kx)$.

This is a straight-forward application of the Euler's formula:

$$\cos^n x = \left(\frac{e^{jx} + e^{-jx}}{2} \right)^n = \frac{1}{2^n} \left(e^{jx} + e^{-jx} \right)^n$$

Now, apply the binomial theorem: $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$. We get

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{jxk} e^{-jx(n-k)} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{jx(2k-n)}$$

Recall that $\binom{n}{n-k} = \binom{n}{k}$ and note that $2(n-k) - n = n - 2k = -(2k-n)$.

So, for every $k = c$, there would be another term $k = n - c$ to pair with. This gives

$$\binom{n}{c} e^{jx(2c-n)} + \underbrace{\binom{n}{n-c} e^{jx(2(n-c)-n)}}_{\binom{n}{c} e^{-jx(2c-n)}} = \binom{n}{c} \times 2 \cos\left((2c-n)x\right)$$

This always happens except when the two terms are actually the same term which occurs when $k = n - k$ or, equivalently, $k = \frac{n}{2}$. In which case,

$$\binom{n}{k} e^{jx(2k-n)} = \binom{n}{n/2} = \binom{n}{n/2} \cos(0x)$$

From the analysis above, we see that $\cos^n(x)$ can be expanded into a linear combination of the cosine.

In particular, $\cos^n(2\pi f_c t)$ can be written as a linear combination of the cosine $\cos(2\pi(2k-n)f_c t)$.

Now, consider $m(t)\cos^n(2\pi f_c t)$. We want to use BPF to extract the content around $\pm f_c$. The content will be there if and only if there is a $\cos(2\pi f_c t)$ term in the expansion of $\cos^n(2\pi f_c t)$.

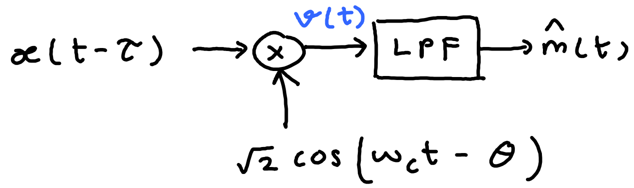
This happens if and only if there is a k value that makes $2k-n=1$.

For a given n, this k value is $k=\frac{n+1}{2}$.

Note that, from the binomial expansion, k must be an integer between 0 and n.

So, n must be odd number to give an integer-valued k.

(a)



where

$$x(t-\tau) = m(t-\tau)\sqrt{2}\cos(\omega_c t - \underbrace{\omega_c \tau}_{\equiv \phi \text{ as defined in lecture.}})$$

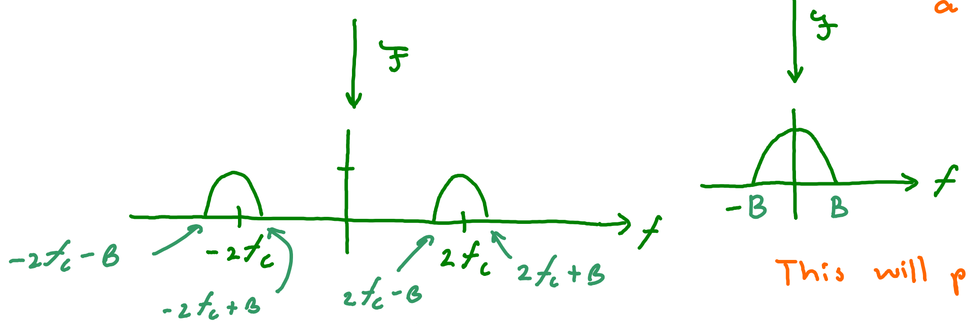
Let $v(t)$ be the signal before the LPF.

Then $v(t) = x(t-\tau) \times \sqrt{2}\cos(\omega_c t - \theta)$

$$= 2\, m(t-\tau)\cos(\omega_c t - \phi)\cos(\omega_c t - \theta)$$

$$= m(t-\tau)\left(\cos(2\omega_c t - \phi - \theta) + \cos(\theta - \phi)\right)$$

$$= m(t-\tau)\cos(2\omega_c t - \phi - \theta) + m(t-\tau)\underbrace{\cos(\theta - \phi)}_{\text{a constant}}$$



This won't pass the LPF.

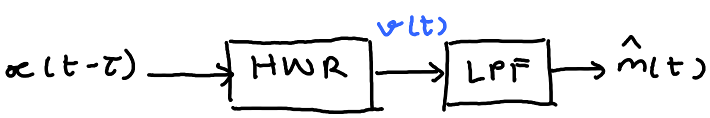
This will pass the LPF.

$$\hat{m}(t) = m(t-\tau)\cos(\theta - \phi) = m(t-\tau)\cos(\theta - \omega_c \tau).$$

(b)

Again, we have

$$x(t-\tau) = m(t-\tau)\sqrt{2}\cos(\omega_c(t-\tau))$$



Let $v(t)$ be the signal before the LPF.

Then, $v(t) = x(t-\tau) \times r(t-\tau)$, where $r(t) = 1[\cos(2\pi f_c t) \geq 0]$

↳ Because $m(t-\tau)$ is always ≥ 0, the sign of $x(t-\tau)$ only depends on $\cos(\omega_c(t-\tau))$, which is simply a shifted version of $\cos(\omega_c t)$.

All of the analysis is the same as what was presented in class except that we now have a time shift of amount τ.

Recall that

$$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos\omega_c t - \frac{2}{\pi}\times\frac{1}{3}\cos 3\omega_c t + \dots$$
$$= \sum_{k=0}^{\infty} a_k \cos(k\omega_c t)$$

where $a_0 = \frac{1}{2}$, $a_1 = \frac{2}{\pi}$, $a_2 = 0$, $a_3 = \frac{2}{\pi}\times\frac{1}{3}$, ...

We then have

$$v(t) = x(t-\tau)\times r(t-\tau)$$
$$= m(t-\tau)\sqrt{2}\cos\left(\omega_c(t-\tau)\right)\sum_{k=0}^{\infty} a_k\cos\left(k\omega_c(t-\tau)\right)$$
$$= \sqrt{2}\, m(t-\tau)\sum_{k=0}^{\infty} a_k \cos\left(\omega_c(t-\tau)\right)\cos\left(k\omega_c(t-\tau)\right)$$
$$= \sqrt{2}\, m(t-\tau)\sum_{k=0}^{\infty}\frac{1}{2}a_k\left(\cos\left((k-1)\omega_c(t-\tau)\right)+\cos\left((k+1)\omega_c(t-\tau)\right)\right)$$

So, $v(t)$ will be a linear combination of signals of the form

$$\sqrt{2}\times\frac{1}{2}\times a_k\times m(t-\tau)\cos\left(\underbrace{n}_{k-1 \text{ or } k+1}\omega_c(t-\tau)\right)$$

We know that the spectrum of $m(t)\cos(n\omega_c t)$ is the spectrum of $m(t)$ shifted to $\pm 2\pi f_c\times n$ and scaled by $\frac{1}{2}$.

The time shift results in an extra factor of $e^{-j2\pi f\tau}$ which does not affect the location of the spectrum.

Recall that $\hat{m}(t) = \text{LPF}\{v(t)\}$.

The only part of $v(t)$ that will pass through the LPF would be the one that is centered around 0 Hz. (DC)

This corresponds to the case when $n=0$

$k-1$ or $k+1$

The corresponding k is $k=1$ or -1.

not in the summation.

Therefore, $\hat{m}(t) = \sqrt{2} \times \frac{1}{2} \times a_1 \times m(t-\tau)$.

For HWR, $a_1 = \frac{2}{\pi}$.

Hence,

$$\hat{m}(t) = \frac{\sqrt{2}}{\pi} m(t-\tau)$$

Q4 FWR Rx with Time Delay

Sunday, August 05, 2012 9:46 PM

(a) Let's start with FWR input-output relation: $f_{FWR}(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$

Here, the input is $x(t-\tau)$. So, the output is $v(t) = \begin{cases} x(t-\tau), & x(t-\tau) \geq 0 \\ -x(t-\tau), & x(t-\tau) < 0. \end{cases}$

Now, we know more about the characteristics of $x(t-\tau)$.

In particular, we know that $x(t-\tau) = m(t-\tau)\cos(\omega_c(t-\tau))$

and that $m(t) \geq 0$ at all t (therefore $m(t-\tau) \geq 0$ at all t.)

The nonnegativity of $m(t)$ means that the sign of $x(t-\tau)$ will depend only on $\cos(\omega_c(t-\tau))$.

Therefore, $v(t) = \begin{cases} x(t-\tau), & \cos(\omega_c(t-\tau)) \geq 0 \\ -x(t-\tau), & \cos(\omega_c(t-\tau)) < 0 \end{cases} = x(t-\tau) \times g_{FWR}(t-\tau)$

where $g_{FWR}(t-\tau) = \begin{cases} 1, & \cos(\omega_c(t-\tau)) \geq 0 \\ -1, & \cos(\omega_c(t-\tau)) < 0. \end{cases}$

In other words, $g_{FWR}(t) = \begin{cases} 1, & \cos(\omega_c t) \geq 0 \\ -1, & \cos(\omega_c t) < 0. \end{cases}$

We have seen in the previous HW question that

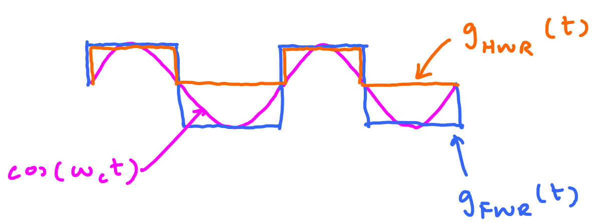
for HWR, $v(t) = x(t-\tau) \times 1[\cos(\omega_c(t-\tau)) \geq 0]$.

So, $v(t) = x(t-\tau) \times g_{HWR}(t-\tau)$

where $g_{HWR}(t) = 1[\cos(\omega_c t) \geq 0]$.

The ON-OFF function.

(i) It is easier to find c_1 and c_2 via the plots of g_{FWR} and g_{HWR}.



From the plots, we have $g_{FWR}(t) = 2g_{HWR}(t) - 1$

Therefore, $c_1 = 2$ and $c_2 = -1$

(ii) $g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi} \sum^{\infty} \frac{(-1)^{k+1}}{} \cos((2k-1)\omega_c t)$

(ii) $g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \cos((2k-1)\omega_c t)$

Therefore, $g_{FWR}(t) = 2 g_{HWR}(t) - 1 = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \cos\left((2k-1)\omega_c t\right)$.

(b) $y(t) = LPF\{v(t)\}$ where $v(t) = m(t-\tau)\cos(\omega_c(t-\tau))\, g_{FWR}(t-\tau)$.

Let's first consider $v(t+\tau) = m(t)\cos(\omega_c t)\, g_{FWR}(t)$.

$$v(t+\tau) = m(t)\cos(\omega_c t)\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \cos((2k-1)\omega_c t)$$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \left(m(t)\cos((2k-2)\omega_c t) + m(t)\cos(2k\,\omega_c t) \right)$$

In freq. domain, these terms will be replica of $M(f)$ shifted to various frequencies.

The only term that shifts to DC is this one at $k=1$.

So, $y(t) = LPF\{v(t)\} = \frac{2}{\pi} m(t-\tau)$.

Q5 AM

Wednesday, October 21, 2015 9:03 PM

Recall that $\mu = \frac{m_p}{A}$ where $m_p = \max_t |m(t)|$

For $m(t) = \alpha \cos(10\pi t)$, $m_p = |\alpha|$.

"A" in the formula above is the same as "A" in this problem.

It is the amplitude of the carrier part of the AM transmitted signal.

So, $\mu = \frac{|\alpha|}{A} = \frac{\alpha}{A}$ Let's consider only $\alpha > 0$ here.

(a) Here, $\alpha = 4 \Rightarrow A = \frac{\alpha}{\mu} = \frac{4}{\mu}$

(i) $A = \frac{4}{0.5} = 8$ (ii) $A = \frac{4}{1} = 4$ (iii) $A = \frac{4}{1.5} = \frac{4}{3/2} = \frac{8}{3}$

(b) Here, $A = 4 \Rightarrow \alpha = A\mu = 4\mu$

(i) $\alpha = 4 \times 0.5 = 2$ (ii) $\alpha = 4 \times 1 = 4$ (iii) $\alpha = 4 \times 1.5 = 4 \times \frac{3}{2} = 6$

Q6 Poisson Sum Formula [M2011 Q5]

Monday, August 06, 2012 7:49 PM

(a) and (b) Recall that $\sum_k \delta(t-kT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta(f-kf_0)$ where $f_0 = \frac{1}{T_0}$.

Of course, you may not remember the above fact. However, I asked you to remember one special case which is much easier to remember:

$$\underbrace{\sum_n \delta(t-n)}_{\text{Let this be } y(t).} \xrightarrow{\mathcal{F}} \sum_k \delta(f-k)$$

The special case can be turned into the general case via the scaling property:

From $y(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}Y\left(\frac{f}{a}\right)$, we have

$$\sum_n \delta(at-n) = \frac{1}{|a|}\sum_n \delta\left(t-\frac{n}{a}\right) \qquad \frac{1}{|a|}\sum_k \delta\left(\frac{f}{a}-k\right) = \frac{|a|}{|a|}\sum_k \delta(f-ak)$$

Recall that $\delta(at) = \frac{1}{|a|}\delta(t)$

Therefore

$$\sum_n \delta\left(t-\frac{n}{a}\right) \xrightarrow{\mathcal{F}} |a|\sum_k \delta(f-ak)$$

Let $a = \frac{1}{T_0}$. We then have $\sum_n \delta(t-nT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta\left(f-\frac{k}{T_0}\right)$

Alternatively, one may always go back to the Fourier series formula to obtain such relationship.

In this question, this property is applied to $\sum_\ell \delta(t-\ell T)$ to get

$$\sum_\ell \delta(t-\ell T) \xrightarrow{\mathcal{F}} \frac{1}{T}\sum_\ell \delta\left(f-\frac{\ell}{T}\right)$$

So, by the convolution-in-time rule, we have $x(t) \xrightarrow{\mathcal{F}} G(f) * \frac{1}{T}\sum_\ell \delta\left(f+\left(-\frac{\ell}{T}\right)\right)$

(c) and (d)

The integral under consideration is $\int_{-\infty}^{\infty} \underbrace{e^{j2\pi f t} G(f)}_{\text{call this } b(f)} \delta\left(t-\frac{\ell}{T}\right) df$

By the sifting property of δ-function,

$$\int_{-\infty}^{\infty} b(f)\delta\left(f-\frac{\ell}{T}\right)df = b\left(\frac{\ell}{T}\right) = e^{j2\pi\frac{\ell}{T}t}\, G\left(\frac{\ell}{T}\right)$$

Summary: $a = \frac{1}{T}$, $b = -\frac{\ell}{T}$, $c = j2\pi\frac{\ell}{T}t$, $d = \frac{\ell}{T}$